# THERMAL CONVECTION IN A ROTATING CIRCULAR PIPE WITH A CONSTANT TEMPERATURE GRADIENT (COMPRESSIBLE FLUID)

## (TEPLOVAIA KONVEKTSIIA VO VRASHCHAIUSHCHEISIA KRUGLOI TRUBE PRI POSTOIANNOM TEMPERATURNOM GRADIENTE (SZHIMAEMAIA ZHIDKOST'))

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In reference [1] it was demonstrated that the equations of thermal convection for an incompressible fluid in an infinitely long rotating pipe become linear when the temperature gradient along the axis of rotation is constant. We now deal with thermal convection in a compressible viscous fluid within an infinitely long circular rotating tube. We deal with the problem in terms of cylindrical coordinates rotating at velocity $\omega$, the polar axis coinciding with the axis of rotation. The conditions of symmetry imply that there is no dependence on $\phi$, and for an infinitely long pipe $\partial v_z/\partial z = 0$, $v_r = v_\phi = 0$. Finally we assume that the fluid obeys the equation of state of an ideal gas. The fluid motion can therefore be described by the following system of equations [2]:

$$\frac{\partial p}{\partial r} = \rho\omega^2 r, \qquad \frac{\partial p}{\partial z} = \eta \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right), \qquad p = \frac{\rho TR}{m} \tag{1}$$

We will assume that temperature variation within the fluid is governed by the effect of a constant temperature gradient at the walls of the tube. Inasmuch as the conditions over a pipe section are similar, we get

$$\frac{\partial T}{\partial z} = a = \text{const} \tag{2}$$

When dealing with the case of a compressible fluid, this single condition for linearizing equation (1) is insufficient, for now it is not possible to neglect density change with change in pressure. To be able to linearize thermal convection equations for a compressible fluid we will study a case of slow convection. That is, we will assume that the temperature variation is small compared to the mean temperature in the fluid, and that the velocity of convection which gives rise to this variation and the changes in both pressure and density are all small; i.e. we can put

$$\begin{gathered} T' = T + \delta T, \quad p' = p + \delta p, \quad \rho' = \rho + \delta\rho, \quad v_z = v \neq 0 \\ \delta T \ll T, \quad \delta p \ll p, \quad \delta\rho \ll \rho, \quad v \ll \omega r_1 \end{gathered} \tag{3}$$

Here $r_1$ is the tube radius and $p$ and $\rho$ are pressure and density respectively at some constant mean temperature $T$ of the fluid. In other words, we are seeking a solution to system (1) in terms of a power series of the gradient $a$, which is assumed to be small. If we keep only the first-order terms, from system (1) it is easy to obtain the following

$$\begin{gathered} \frac{\partial p}{\partial r} + \frac{\partial \delta p}{\partial r} = p\frac{m\omega^2 r}{RT} + \frac{m\omega^2 r}{RT}\left(\delta p - p\frac{\delta T}{T}\right) \\ \frac{\partial p}{\partial z} + \frac{\partial \delta p}{\partial z} = \eta \frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) \end{gathered} \tag{4}$$

For the isothermal case $\delta T = 0$, $\delta p = 0$, $v = 0$, and the fluid rotates as a whole with pressure distribution given by:

$$p = p_0 \exp\frac{m\omega^2 r^2}{2RT} \tag{5}$$

where $p_0$ is the pressure at the tube axis. Now we neglect $\delta p$ in equation (4), and putting in (2) and (5) we obtain an equation for slow convection

$$\frac{m\omega^2 r}{RT}\left[\eta\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) - \frac{ap_0}{T}\exp\frac{m\omega^2 r^2}{2RT}\right] = \eta\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right)\right] \tag{6}$$

This equation should be solved with the following boundary conditions

(1) finiteness of solution for $r = 0$

(2) $$v(r_1) = 0 \tag{7}$$

(3) $$2\pi\int_0^{r_1} \rho v r\, dr = \pi r_1^2 \rho Q$$

where $Q$ is the given mass flow of fluid in the tube.

We now introduce the following variables and definitions

$$x = \frac{m\omega^2 r^2}{2RT}, \quad b = \frac{m\omega^2 r_1^2}{2RT}, \quad v = \frac{ar_1^2 p_0}{4b\eta T}u, \quad Q = \frac{ar_1^2 p_0}{4b^2\eta T}q \tag{8}$$

Hence equation (6) and boundary conditions (7) become

$$\frac{d^2}{dx^2}\left(x\frac{du}{dx}\right) - \frac{d}{dx}\left(x\frac{du}{dx}\right) + e^x = 0, \quad u(b) = 0, \quad \int_0^b e^x u\, dx = q \tag{9}$$

It is easy now to obtain

$$u = [(e^b - 1)^2 - 2q]\frac{J_1(x) - J_1(b)}{2J_2(b) - 4J_1(b)} + e^b - e^x \qquad (10)$$

where

$$J_1(x) = \int_0^x \frac{e^x - 1}{x}\,dx, \qquad J_2(b) = \int_0^b \frac{e^{2x} - 1}{x}\,dx$$

Or, finally,

$$v = \frac{ar_1^2 p_0}{4b\eta T}\left\{\frac{(e^b - 1)^2}{2\,[\mathrm{Ei}\,(2b) - 2\,\mathrm{Ei}\,(b) + \ln\gamma b - \ln 2]}\left[\mathrm{Ei}\,(x) - \mathrm{Ei}\,(b) - \ln\frac{x}{b}\right] + e^b - e^x\right\} + \\ + \frac{bQ}{\mathrm{Ei}\,(2b) - 2\,\mathrm{Ei}\,(b) + \ln\gamma b - \ln 2}\left[\mathrm{Ei}\,(b) - \mathrm{Ei}\,(x) + \ln\frac{x}{b}\right] \qquad (11)$$

where $\mathrm{Ei}(x)$ is a function tabulated in (3).

This solution is composed of the superposition of two pressures; the first term in (11) expresses free convection in the compressible rotating fluid, and the second is the forced motion of the compressible fluid in the rotating pipe due to the difference in external pressure (similar to poiseuille flow of incompressible fluid).

## BIBLIOGRAPHY

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